Chapter 6

Elliptic Equations

Finite-Difference Method

Second Session Contents:

1) **Incompressible Flow Using Stream Function-Vorticity**

Stream Function-Vorticity

Stream Function-Vorticity Formulation is suitable for

2-D incompressible flows and axis symmetry geometries

Applications:

- Laminar and Turbulence flows
- Transition
- Free convection flows
- Mixed convection flows

Stream Function-Vorticity

Advantages

- It is not necessary to solve pressure field.
- In many algorithms, conservation of mass is solved to find the pressure field, but in this method, continuuity is removed from the equations.
- The pressure terms are removed from the navier-stokes equations using cross derivatives.
- The obtained equations are Elliptic for low Reynolds number.

Stream Function-Vorticity

Disadvantages

- This method cannot be applied on 3-D problems.
- Additional calculations are necessary to obtain the pressure field from the numerical results.
- The vorticity boundary condition should be specified iteratively from the obtained stream functions.
- The order of solution can be influenced from the order of boundary conditions treatment.

Lid-Driven Cavity

Governing Equations

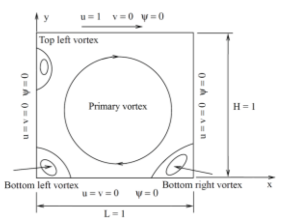
$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\omega$$

ψ: Stream function

ω: Vorticity

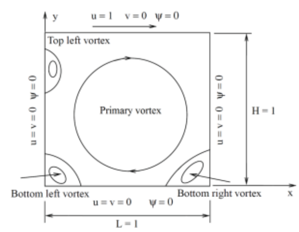
$$u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

$$u=\frac{\partial\psi}{\partial y}\qquad v=-\frac{\partial\psi}{\partial x}$$



Boundary Conditions (No-slip condition)

$$u = Uo,\quad v = 0\quad at\ \ y = H$$
$$u = 0,\quad v = 0\quad at\ \ y = 0$$
$$u = 0,\quad v = 0\quad at\ \ x = l$$
$$u = 0,\quad v = 0\quad at\ \ x = 0$$



Boundary Conditions (Stream Function)

$$u=\frac{\partial\psi}{\partial y}=0\qquad x=0,\qquad x=l$$

ψ is constant on the left and right walls

$$v=-\frac{\partial\psi}{\partial x}=0\qquad y=0,\qquad y=H$$

ψ is constant on the top and bottom walls

Boundary Conditions (Stream Function)

$$u=\frac{\partial\psi}{\partial y}=0\qquad x=0,\qquad x=l$$

ψ is constant on the left and right walls

$$v=-\frac{\partial\psi}{\partial x}=0\qquad y=0,\qquad y=H$$

ψ is constant on the top and bottom walls

ψ is constant on all walls

Boundary Conditions (Stream Function)

$u = \frac{\partial \psi}{\partial y} = 0 \qquad x = 0, \qquad x = l$

ψ is constant on the left and right walls

$v = -\frac{\partial \psi}{\partial x} = 0 \qquad y = 0, \qquad y = H$

ψ is constant on the top and bottom walls

ψ is constant on all walls

$\psi = 0$	$x = 0$	$0 \le y \le H$
$\psi = 0$	$x = l$	$0 \le y \le H$
$\psi = 0$	$y = 0$	$0 \le x \le l$
$\psi = 0$	$y = 0$	$0 \le x \le l$

Boundary Conditions (Vorticity)

$\psi = \text{const.}$
$x = 0\,, \qquad x = l$

$\frac{\partial \psi}{\partial y} = \frac{\partial^2 \psi}{\partial y^2} = 0$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \rightarrow \frac{\partial^2 \psi}{\partial x^2} = -\omega$$

$\psi = \text{const.}$
$y = 0\,, \qquad y = H$

$\frac{\partial \psi}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} = 0$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \rightarrow \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

Discretization

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

$$\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} = -\omega_{i,j}$$

$$\psi_{i+1,j} + \psi_{i-1,j} - 2(1+\beta^2)\psi_{i,j} + \beta^2(\psi_{i,j+1} + \psi_{i,j-1}) = -\Delta x^2 \omega_{i,j}$$

$\psi_{1,j} = 0 \qquad 1 \le j \le jmax \qquad \psi_{imax,j} = 0 \qquad 1 \le j \le jmax$

$\psi_{i,1} = 0 \qquad 1 \le i \le imax \qquad \psi_{i,jmax} = 0 \qquad 1 \le i \le imax$

Discretization

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

$$u_{i,j}\frac{\omega_{i+1,j} - \omega_{i-1,j}}{2\Delta x} + v_{i,j}\frac{\omega_{i,j+1} - \omega_{i,j-1}}{2\Delta y} = \nu\left\{\frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{\Delta x^2} + \frac{\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}}{\Delta y^2}\right\}$$

$$u_{i,j} = \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \qquad v_{i,j} = -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x}$$

Discretization

$$\omega_{i+1,j} + \omega_{i-1,j} - 2(1+\beta^2)\omega_{i,j} + \beta^2(\omega_{i,j+1} + \omega_{i,j-1}) - \frac{u_{i,j}\Delta x}{2\nu}(\omega_{i+1,j} - \omega_{i-1,j}) - \frac{\beta v_{i,j}\Delta x}{2\nu}(\omega_{i,j+1} - \omega_{i,j-1}) = 0$$

$\beta = \Delta x/\Delta y$

$2 \le i \le imax - 1$

$2 \le j \le jmax - 1$

$$\frac{\partial^2\psi}{\partial x^2} = -\omega \qquad \omega_{1,j} = -\frac{\psi_{2,j} + \psi_{0,j} - 2\psi_{1,j}}{\Delta x^2}$$

Discretization (Boundary Conditions)

$$\omega_{i+1,j} + \omega_{i-1,j} - 2(1+\beta^2)\omega_{i,j} + \beta^2(\omega_{i,j+1} + \omega_{i,j-1}) - \frac{u_{i,j}\Delta x}{2\nu}(\omega_{i+1,j} - \omega_{i-1,j}) - \frac{\beta v_{i,j}\Delta x}{2\nu}(\omega_{i,j+1} - \omega_{i,j-1}) = 0$$

$\beta = \Delta x/\Delta y$

$2 \le i \le imax - 1$

$2 \le j \le jmax - 1$

$$\frac{\partial^2\psi}{\partial x^2} = -\omega \qquad \omega_{1,j} = -\frac{\psi_{2,j} + \psi_{0,j} - 2\psi_{1,j}}{\Delta x^2}$$

$$\left.\frac{\partial\psi}{\partial x}\right|_{1,j} = \frac{\psi_{2,j} - \psi_{0,j}}{2\Delta x} = 0 \longrightarrow \psi_{0,j} = \psi_{2,j}$$

$$\omega_{1,j} = -\left\{\frac{2\psi_{2,j} - 2\psi_{1,j}}{\Delta x^2}\right\} = -\frac{2\psi_{2,j}}{\Delta x^2}$$

Discretization (Boundary Conditions)

$$\omega_{1,j} = -\left\{\frac{2\psi_{2,j} - 2\psi_{1,j}}{\Delta x^2}\right\} = -\frac{2\psi_{2,j}}{\Delta x^2} \qquad 2 \le j \le jmax - 1$$

$$\omega_{i,1} = -\frac{2\psi_{i,2}}{\Delta y^2} \qquad 2 \le i \le imax - 1$$

$$\omega_{imax,j} = -\frac{2\psi_{imax-1,j}}{\Delta x^2} \qquad 2 \le j \le jmax - 1$$

$$\omega_{i,jmax} = -\frac{\partial^2\psi}{\partial y^2} = -\frac{\psi_{i,jmax-1} + \psi_{i,jmax+1} - 2\psi_{i,jmax}}{\Delta y^2} = -\frac{(\psi_{i,jmax-1} + \psi_{i,jmax+1})}{\Delta y^2}$$

$$u = \frac{\partial\psi}{\partial y} = \frac{\psi_{i,jmax+1} - \psi_{i,jmax-1}}{2\Delta y} = U_0 \qquad \omega_{i,jmax} = -\frac{(2\psi_{i,jmax-1} + 2U_0\Delta y)}{\Delta y^2}$$

Stream Function–Vorticity

- Each equation has five unknowns
- Using iterative line-by-line method and therefore TDMA is suggested.

To have a stable solution, the following condition should be satisfied

$$\left|\frac{u_{i,j}\Delta x}{\nu}\right| \le 2 \qquad \left|\frac{v_{i,j}\Delta y}{\nu}\right| \le 2$$

Stream Function–Vorticity

Initial Guess

$$\psi_{i,j} = 0 \qquad \omega_{i,j} = -\frac{j}{jmax}\frac{2U_\circ}{\Delta y}$$

Relaxation Factor

$$\psi_{i,j}^{k+1} = \psi_{i,j}^{k} + W_s(\psi_{i,j}^{*} - \psi_{i,j}^{k})$$
$$\omega_{i,j}^{k+1} = \omega_{i,j}^{k} + W_v(\omega_{i,j}^{*} - \omega_{i,j}^{k})$$

Upwind Method

- Using central difference method, the grid step size is limited. Therefore, solution is stable only for low Reylods numbers and computational costs increases.
- To achieve this difficulties, the Upwind method can be used. So, the solution is stable both for large grid step size and high Reynolds numbers.
- In upwind method, the convection term are discretized based on direction of wind.

Upwind Method

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = u_{i,j}\left(\frac{\omega_{i,j} - \omega_{i-1,j}}{\Delta x}\right) + v_{i,j}\left(\frac{\omega_{i,j} - \omega_{i,j-1}}{\Delta y}\right) \qquad u_{i,j} > 0, v_{i,j} > 0$$

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = u_{i,j}\left(\frac{\omega_{i+1,j} - \omega_{i,j}}{\Delta x}\right) + v_{i,j}\left(\frac{\omega_{i,j} - \omega_{i,j-1}}{\Delta y}\right) \qquad u_{i,j} < 0, v_{i,j} > 0$$

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = u_{i,j}\left(\frac{\omega_{i,j} - \omega_{i-1,j}}{\Delta x}\right) + v_{i,j}\left(\frac{\omega_{i,j+1} - \omega_{i,j}}{\Delta y}\right) \qquad u_{i,j} > 0, v_{i,j} < 0$$

$$u\frac{\partial\omega}{\partial x} + v\frac{\partial\omega}{\partial y} = u_{i,j}\left(\frac{\omega_{i+1,j} - \omega_{i,j}}{\Delta x}\right) + v_{i,j}\left(\frac{\omega_{i,j+1} - \omega_{i,j}}{\Delta y}\right) \qquad u_{i,j} < 0, v_{i,j} < 0$$

Hybrid Method

Using both upwind and central difference method we have

$$\omega_{i+1,j} + \omega_{i-1,j} - 2(1+\beta^2)\omega_{i,j} + \beta^2(\omega_{i,j+1} + \omega_{i,j-1})$$
$$-\frac{u_{i,j}\Delta x}{\nu}(\mathrm{cxp} * \omega_{i+1,j} + \mathrm{cxo} * \omega_{i,j} + \mathrm{cxm} * \omega_{i-1,j})$$
$$-\frac{\beta v_{i,j}\Delta x}{\nu}(\mathrm{cyp} * \omega_{i,j+1} + \mathrm{cyo} * \omega_{i,j} + \mathrm{cym} * \omega_{i,j-1}) = 0$$

Hybrid Method

$\text{cxp} = 1/2,\ \text{cxo} = 0,\ \text{cxm} = -1/2 \qquad \left|\dfrac{u_{i,j}\Delta x}{\nu}\right| \le 2$ Central Difference

$\text{cyp} = 1/2,\ \text{cyo} = 0,\ \text{cym} = -1/2 \qquad \left|\dfrac{v_{i,j}\Delta y}{\nu}\right| \le 2$ Central Difference

$$\begin{cases} \text{cxp} = 0, & \text{cxo} = 1, & \text{cxm} = -1 & u > 0 \\ \text{cxp} = 1, & \text{cxo} = -1, & \text{cxm} = 0 & u < 0 \end{cases} \qquad \left|\frac{u_{i,j}\Delta x}{\nu}\right| > 2$$

$$\begin{cases} \text{cyp} = 0, & \text{cyo} = 1, & \text{cym} = -1 & v > 0 \\ \text{cyp} = 1, & \text{cyo} = -1, & \text{cym} = 0 & v < 0 \end{cases} \qquad \left|\frac{v_{i,j}\Delta y}{\nu}\right| > 2$$

Convection Terms

- Upwind $O(h)$
- Central $O(h^2)$

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Truncation Error

$$\psi_{2,j} = \psi_{1,j} + \left.\frac{\partial \psi}{\partial x}\right|_{1,j} \Delta x + \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2!} + \left.\frac{\partial^3 \psi}{\partial x^3}\right|_{1,j} \frac{\Delta x^3}{3!} + \left.\frac{\partial^4 \psi}{\partial x^4}\right|_{1,j} \frac{\Delta x^4}{4!} + O(\Delta x^5)$$
$$= \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{1,j} \frac{\Delta x^2}{2!} + \left.\frac{\partial^3 \psi}{\partial x^3}\right|_{1,j} \frac{\Delta x^3}{3!} + \left.\frac{\partial^4 \psi}{\partial x^4}\right|_{1,j} \frac{\Delta x^4}{4!} + O(\Delta x^5)$$

Substituting the second derivatives, we have

$$\omega_{1,j} = \frac{-2\psi_{2,j}}{\Delta x^2} + \left.\frac{\partial^3 \psi}{\partial x^3}\right|_{1,j} \frac{\Delta x}{3} + \left.\frac{\partial^4 \psi}{\partial x^4}\right|_{1,j} \frac{\Delta x^2}{12} + O(\Delta x^3)$$

The truncation error on the boundaries is in order of $\boldsymbol{O}(\Delta x)$

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Truncation Error

$$\frac{\partial^2 \psi}{\partial x^2} = -\omega \qquad \left.\frac{\partial \omega}{\partial x}\right|_{x=0} = -\left[\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial^2 y}\right]_{x=0} = -\left.\frac{\partial^3 \psi}{\partial x^3}\right|_{x=0}$$

$$\omega_{1,j} = \frac{-2\psi_{2,j}}{\Delta x^2} - \left.\frac{\partial \omega}{\partial x}\right|_{1,j} \frac{\Delta x}{3} + \left.\frac{\partial^4 \psi}{\partial x^4}\right|_{1,j} \frac{\Delta x^2}{12} + O(\Delta x^3)$$
$$= \frac{-2\psi_{2,j}}{\Delta x^2} - \left\{\frac{(\omega_{2,j} - \omega_{1,j})}{\Delta x} + O(\Delta x)\right\} \frac{\Delta x}{3} + \left.\frac{\partial^4 \psi}{\partial x^4}\right|_{1,j} \frac{\Delta x^2}{12} + O(\Delta x^3)$$

$$\omega_{1,j} = -\frac{3\psi_{2,j}}{\Delta x^2} - \frac{1}{2}\omega_{2,j} + O(\Delta x^2)$$

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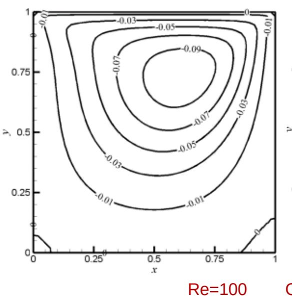
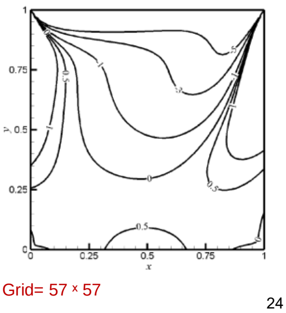
Results

Contours of Stream function



Contours of Vorticity



Re=100 Grid= 57 × 57

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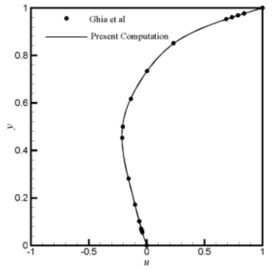
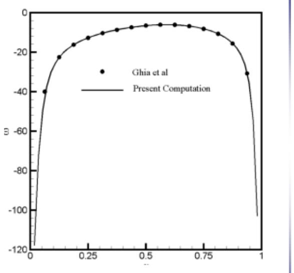
Results



u at $x = 0.5$



ω on the top wall

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